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Momentum conservation forbids sharp localisation

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Abstract. It is well known that conservation laws impose limitations on the measurability of quantum mechanical observables. In particular, it has been shown that predictable and repeatable position measurements are impossible due to momentum conservation. Here we provide evidence that the assumption of repeatability may even be dropped. The consequences of this result for the concept of observable and of quantum theoretical reality are discussed elsewhere.

1. Introduction

Quantum mechanical observables have been represented by self-adjoint operators or, equivalently, by their spectral measures. It is well known that not all self-adjoint operators are observables; this is excluded for example by superselection rules. Furthermore, in the development of a physical theory new laws may be discovered which impose restrictions on the measurability of certain 'observables'. In particular, once the theory is rich enough to include a description of measurement as physical process one should check within a theory of measurement whether or not the 'measurable quantities' (observables) are indeed 'observable'. Wigner (1952) showed that repeatable and predictable spin measurements are impossible due to conservation of angular momentum. The idea is strikingly simple: assume a unitary evolution operator U obeying

$$U(\varphi_\pm \otimes \chi_0) = \varphi_\pm \otimes \chi_\pm \qquad (\chi_+, \chi_-) = 0.$$

($\varphi_\pm$ are the eigenstates of s_z, the z component of spin, χ_0 and $\chi_\pm$ are the initial and final states of the measuring device, respectively.) Then the x component of angular momentum of the combined system cannot be conserved; this can immediately be seen from the equations

$$U[(\varphi_+ \pm \varphi_-) \otimes \chi_0] = \varphi_+ \otimes \chi_+ \pm \varphi_- \otimes \chi_-$$

where the x-spin values are the same on the right-hand sides but not on the left-hand sides.

This result has been generalised in various respects by Araki and Yanase (1960) and by Shimony and Stein (1971): in short, if an observable M admits repeatable and predictable measurements, and if $L \otimes I + I \otimes L_A$ is a conserved quantity for the combined object and apparatus system then M commutes with L. More explicitly, if boundedness of L is assumed then the statement applies to *arbitrary repeatable* measurements:

$$U[E_m \otimes \langle \chi_0 \rangle] \subseteq E_m \otimes F_m$$

(here E_m are pairwise orthogonal M spectral spaces in the apparatus state space); i.e., arbitrary state changes within an M eigenspace can be admitted. On the other hand, if boundedness of L is not required then only 'finitely distorting' measurements in the sense described by Shimony and Stein (for example, measurements satisfying the projection postulate) can be taken into account. Only the second version with its comparably weak claim is applicable to unbounded conserved quantities as momentum. In this paper which was inspired mainly by the work of Shimony and Stein the question of (repeatable) position measurements in view of momentum conservation is investigated within a model. We shall show that, under reasonable assumptions, *arbitrary predictable* position measurements—whether repeatable or not—are forbidden due to conservation of momentum.

2. A theorem

We shall prove the following theorem.

Theorem. Let $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{L}^2(\mathbb{R})$, $\mathscr{H}_1^+ = \mathscr{L}^2(0, \infty)$, $\mathscr{H}_1^- = \mathscr{L}^2(-\infty, 0)$, $\eta_0 \in \mathscr{H}_2$, U a unitary operator on $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, $P = P_1 \otimes I + I \otimes P_2$ the closed momentum (derivation) operator on $\mathscr{H}$. Then U and P do not commute if (I) or (II) is fulfilled:

(I) There exist η_+, η_- in $\mathscr{H}_2$, $(\eta_+, \eta_-) = 0$, such that for all $\varphi_\pm$ in $\mathscr{H}_1^\pm$ there are $\varphi'_\pm$ in $\mathscr{H}_1$ with

$$U(\varphi_\pm \otimes \eta_0) = \varphi'_\pm \otimes \eta_\pm.$$

(II) $\operatorname{supp}(\eta_0) \subseteq [-a, a]$; for all $\varphi_\pm$ in $\mathscr{H}_1^\pm$ there exist $\varphi'_\pm$ in $\mathscr{H}_1$, $\varphi'_\pm = \varphi'_\pm[\varphi_\pm]$, $\eta_\pm = \eta_\pm[\varphi_\pm]$ in $\mathscr{H}_2$ with $\operatorname{supp}(\eta_+) \subseteq [a, \infty]$, $\operatorname{supp}(\eta_-) \subseteq [-\infty, -a]$ such that

$$U(\varphi_\pm \otimes \eta_0) = \varphi'_\pm \otimes \eta_\pm.$$

Proof. We assume commutativity of U and P; i.e., for all real q, $U \exp(iqP) - \exp(iqP)U = 0$. Denote $V_q := \exp(iqP) = \exp[iq(P_1 \otimes I + I \otimes P_2)] = W_q \otimes W_q$, $\psi^q := W_q\psi$, $\psi^q(x) = \psi(x+q)$ for ψ in $\mathscr{H}_i$ $(i = 1, 2)$. Then consider the scalar product of $V_q\varphi_+ \otimes \eta_0 = \varphi_+^q \otimes \eta_0^q$ with $\varphi_- \otimes \eta_0$

$$(V_q\varphi_+ \otimes \eta_0, \varphi_- \otimes \eta_0) = (\varphi_+^q, \varphi_-)(\eta_0^q, \eta_0).$$

From unitarity of U and commutativity of U and V_q we obtain

$$(\varphi_+^q, \varphi_-)(\eta_0^q, \eta_0) = (\varphi_+'^q, \varphi'_-)(\eta_+^q, \eta_-) \qquad \forall q \in \mathbb{R}. \tag{$*$}$$

Now we proceed with part (I): continuity of the map $q \to V_q$ implies that, for all real $\varepsilon > 0$, there exists real $\delta > 0$ such that

$$|(\eta_0^q, \eta_0)| > 1 - \varepsilon \qquad \text{and} \qquad |(\eta_+^q, \eta_-)| < \varepsilon \qquad \text{for} \qquad |q| < \delta.$$

We may assume $\|\varphi_+\| = \|\varphi_-\| = \|\eta_0\| = 1$. Then unitarity of U gives

$$N := \|\eta_+\| \cdot \|\eta_-\| = (\|\varphi'_+\| \cdot \|\varphi'_-\|)^{-1}.$$

Now choose φ_+, φ_- such that, for some q with $|q| < \delta$, $\varphi_+^q = \varphi_-$. Then ($*$) implies $(\varphi_+'^q, \varphi'_-) \neq 0$ and $(1 - \varepsilon) < \varepsilon \|\varphi'_+\| \cdot \|\varphi'_-\| = \varepsilon / N$, i.e. $N/(N+1) < \varepsilon$, in contradiction to the arbitrariness of ε. This proves (I).

To show part (II), take $|q| \neq 0$ small enough such that $(\eta_0^q, \eta_0) \neq 0$ and $(\eta_+^q, \eta_-) = 0$ $(|q| < 2a)$. Further take φ_+, φ_- with $(\varphi_+^q, \varphi_-) \neq 0$. Then the left-hand side of ($*$) is non-zero whereas the right-hand side is zero. Thus U and P cannot commute.

Before turning to the physical implications some remarks are in order. The first part of our theorem rests on the somewhat restrictive assumption that the apparatus final states $\eta_\pm$ are the same for all object initial states $\varphi_\pm$ in $\mathscr{H}_1^\pm$, respectively. This restriction is relaxed in the second part where we introduce the rather natural requirement that the 'pointer positions' of the apparatus should be well distinguishable from each other: the zero (η_0) as well as the positive (η_+) and negative (η_-) positions are spatially separated (as is practically the case in real devices). Conditions (I) and (II) represent two types of position measurement procedures conflicting with momentum conservation. It is remarkable that no assumption on the object final states is necessary as was the case in the cited works (with the exception of the spin-$\frac{1}{2}$ case). In particular, 'repeatability' ($\varphi'_\pm$ in $\mathscr{H}_1^\pm$) need not be postulated; we only apply the property of 'predictability' (probability one for $\eta_\pm$ if $\varphi_\pm$ in $\mathscr{H}_1^\pm$). This gain has been paid by additional though plausible assumptions on the measuring apparatus. To get our result we reversed the proof direction of the previous authors who inferred commutativity of M and L from conservation laws. Here we start with the very structures of Q and P to show that U and P do not commute.

3. Physical implications

Our model result represents a strong indication that the usual notion of position observable (imprimitivity system for the Euclidean group) must be given up: predictability, as represented by the spectral projections of Q, cannot be realised experimentally for principal reasons. As in the case of discrete bounded observables (see Wigner 1952, Araki and Yanase 1960, Ghirardi *et al* 1982), it can be seen (in the models by Busch (1985a, b)) that deviations from certainty can be made appreciably small by taking 'large' measuring devices. Thus it remains *practically* justified to employ the familiar position spectral measure for the description of localisation. However, from a *theoretical* point of view it is necessary to establish manifest consistency between concepts of physical language and physical laws formulated in that language. The question of which kind of localisation observable *can* be measured has been treated in a model in Busch (1985a). It turns out that so-called systems of covariance—certain positive operator valued measures—are the appropriate tools for dealing with localisation in quantum mechanics (cf the monograph by Prugovecki 1984):

$$X \mapsto Q_f(X) := (\chi_X * f)(Q) \qquad X \in \mathscr{B}(\mathbb{R}) \qquad f \in \mathscr{L}^1(\mathbb{R}).$$

These translation covariant POV measures which are compatible with momentum conservation correspond to measurements yielding only unsharp, smeared-out position values as indicated by the above generalised characteristic function $\chi_X * f$. (See Busch (1985b) for a detailed measurement theoretic interpretation of unsharp observables.) Unsharpness implies that there is (usually) no object state admitting *certain* predictions about some position interval. It should be remarked that in a somewhat different treatment of measurement Schroeck (1985) arrives at a similar result: certain 'stochastic' observables admit measurements consistent with conservation laws.

Position is one of the most fundamental observables; it can be argued that the introduction of any observable rests on position determination. Impossibility of position predictability then gives rise to lack of predictability in the case of *all* observables. This is exemplified in the spin case in Prugovecki (1977) and Schroeck (1982). A concept of unsharp reality has been developed in Busch (1985a) as an extension of the famous Einstein–Podolsky–Rosen reality criterion to provide an adequate physical interpretation of stochastic, or unsharp observables: although the loss of predictability forbids speaking of *properties* of quantum systems (elements of reality) there exists a consistent notion of unsharp reality, or *unsharp properties* of systems.

4. Outlook

Shimony and Stein (1979) formulated the problem of repeatable position measurements in view of momentum conservation in terms of quite a similar model as ours. Our premises imply their assumptions; thus they are stronger. The main difference is that we demand spatial separation not only of the apparatus final states η_+, η_- but of η_+, η_-, and η_0. However, our conclusion is also remarkably stronger: not only repeatable, but *arbitrary predictable* measurements are excluded. Thus we are left with the problem whether the assumption about η_0 may be dropped.

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